We note that the arguments presented above refer to the case of "strong" nonuniformity, when the flow rate changes in the entire region. The local disturbance of the flow rate in the end region of the gap, where the gradients are largest, occurs under less rigid conditions [8].

NOTATION

h, width of the channel; L, length of the channel; $\delta = h/L$, relative width of the channel; ℓ, thickness of the walls; $\beta = \ell/L$, relative thickness of the walls; p, gas pressure; p_e, saturation pressure; $\pi = p_e - p$, pressure drop; T, temperature; T_0, temperature of the evaporation surface; Q, heat of evaporation; $\bar{Q} = Q/RT$, dimensionless heat of evaporation; μ, viscosity of the gas; λ, thermal conductivity of the gas; λ_T, thermal conductivity of the solid phase; q, heat flux; J, mass flow rate; a, velocity of sound; M, Mach's number; Re, Reynold's number; Kn, Knudsen number; and H_π and H_p, dimensionless complexes.

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LIMITING SOLUTION OF A DIFFUSIONAL PROBLEM IN PRISMATIC TUBES

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The limiting solution of the convective-diffusion equation is investigated in channels close to the tube "axis," i.e., the line at which the liquid flow rate takes on a maximum value.

The solution of heat- and mass-transfer problems in prismatic tubes, even when the tubes are linear, entails well-known difficulties when using the methods of mathematical physics [1]. In connection with this, approximate methods are widely used: numerical methods [2], variational and projectional methods [3], methods based on introducing an effective (Taylor) diffusion coefficient [4], and various modifications and improvements of these [5-7].

In the present work, small-perturbation theory is used to investigate a characteristic solution of the problem of impurity propagation in prismatic tubes at large Peclet numbers. The behavior of the impurity concentration around the tube axis is of interest here. The liquid is assumed to be Newtonian and the liquid flow to be laminar, although the individual assumptions of the theory and calculations may simply be extended to more complex cases.

I. Plane Channel

Suppose that the liquid is of sufficiently high viscosity that the liquid flow is stabilized over time and, at the same time, the diffusional process is unstable. The impurity-diffusion equation in this case takes the form

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$$\partial C/\partial t + u\,(1 - Y^2/h^2)\,\partial C/\partial Z = D\partial^2 C/\partial Y^2, \qquad (1)$$

and here, as is often the case, the terms $\partial^2C/\partial Z^2$ are neglected in comparison with $\partial^2C/\partial Y^2$. The additional conditions assumed are as follows

$$C|_{t=0} = 0, \quad C|_{Z=0} = C_0, \quad \partial C/\partial Y|_{Y=0,\pm h} = 0. \qquad (2)$$

Reducing Eqs. (1) and (2) to dimensionless form by the transformation

$$t = \tau h/u, \quad Z = zh, \quad Y = yh, \quad C = cC_0, \quad \varepsilon = 1/\mathrm{Pe} = D/uh \qquad (3)$$

gives

$$\partial c/\partial\tau + (1 - y^2)\,\partial c/\partial z = \varepsilon\partial^2 c/\partial y^2, \qquad (4)$$

$$c|_{\tau=0} = 0, \quad c|_{z=0} = 1, \quad \partial c/\partial y|_{y=0,\pm 1} = 0. \qquad (5)$$

If ε tends to zero in Eq. (4), the equation of the external problem is obtained

$$\partial c/\partial\tau + (1 - y^2)\,\partial c/\partial z = 0, \qquad (6)$$

with the following solution satisfying the first two conditions in Eq. (5)

$$c = H\,[\tau - z/(1 - y^2)], \qquad (7)$$

where H(z) is the Heaviside function. The diffusion coefficient drops out of the solution obtained completely. However, at sufficiently large times, when the region of nonzero concentration bounded by the parabola $z = \tau(1 - y^2)$ becomes thin close to the vertex of the parabola, it is incorrect to neglect the diffusional term. The physical picture will correspond to the propagation of a diffusional jet along the channel axis. Correct description of the process at such times entails considering the internal problem [8], while retaining both diffusional and convective terms in the equation.

Passing to the coordinate system $\tau = \tau$, $\xi = \tau - z$ which moves at velocity u, and then to the internal coordinates $\eta = y/\varepsilon^{1/4}$, $T = \tau\varepsilon^{1/2}$, the parameter ε "disappears" from Eq. (4)

$$\partial c/\partial T + \eta^2\partial c/\partial\xi = \partial^2 c/\partial\eta^2. \qquad (8)$$

Equation (8) determines the diffusional process close to the channel axis. Additional conditions are obtained from the third condition of Eq. (5) and by means of matching with the solution of the external problem in Eq. (7)

$$\partial c/\partial\eta|_{\eta=0} = 0, \quad c|_{\eta\to\infty} \to 0, \quad c|_{T=0} = 1, \quad c|_{\xi=0} = 0. \qquad (9)$$

We consider ξ in the region $(0, \infty)$. When $\xi < 0$, the solution of Eq. (8) is identically zero. In view of the problem's symmetry relative to the line $\eta = 0$, η is assumed to be varying in the region $(0, \infty)$. Note that Eq. (8) is intermediate in character, i.e., decribes the local solution of the initial problem in the time range $O(1/\varepsilon^{1/2}) < \tau < O(1/\varepsilon)$, where the lower bound is determined by the transformation $T = \tau\varepsilon^{1/2}$ and the upper bound by the condition of diffusional "spreading" of the initial concentration perturbation over the whole region U (it is sufficient that the boundaries $Y = \pm h$ exert a considerable influence on the solution fo the internal problem).

A Laplace transformation with respect to the variable ξ is applied to Eq. (8) and the additional condition in Eq. (9) (quantities in the space of the mappings are denoted by an asterisk)

$$\partial c^*/\partial T + p\eta^2 c^* = \partial^2 c^*/\partial\eta^2, \qquad (10)$$

$$\partial c^*/\partial\eta|_{\eta=0} = 0, \quad c^*|_{\eta\to\infty} \to 0, \quad c^*|_{T=0} = 1/p. \qquad (11)$$

Equation (10) admits of variable separation. The solution of Eqs. (10) and (11) obtained by the Fourier method is

$$c^* = \frac{\sqrt{2}}{p}\sum_{n=0}^{\infty} \exp\,[-(4n+1)\,Tp^{1/2} - \eta^2 p^{1/2}/2]\,H_{2n}(\eta p^{1/4})/4^n n! =$$
$$= \frac{\sqrt{2}}{p}\exp\left[-T\sqrt{p} + \frac{\eta^2\sqrt{p}}{2} - \frac{\eta^2\sqrt{p}}{1 + \exp(-4T\sqrt{p})}\right] \Big/ [1 + \exp(-4T\sqrt{p})]^{1/2}, \qquad (12)$$

where $H_{2n}(z)$ are Hermite polynomials.

The total amount of impurity in some cross section ξ is now found, by integrating Eq. (12) with respect to η over the limits $(0, \infty)$

$$Q^* = \int_0^\infty c^* d\eta = \frac{(\pi/2)^{1/2}}{p^{5/4}[\operatorname{sh}(2T\sqrt{p})]^{1/2}} = \frac{\sqrt{\pi}}{p^{5/4}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{\exp[-T\sqrt{p}(4n+1)]}{4^n}. \tag{13}$$

Term-by-term manipulation of the series in Eq. (13) leads to the need to calculate an integral of the form

$$K(x, y) = \frac{1}{2\pi i} \int_L \frac{\exp(px - y\sqrt{p})}{p^{5/4}} dp, \tag{14}$$

where the integration is taken over the straight line Re $p = \alpha$, $\alpha > 0$. It is readily evident that the function K satisfies the heat-conduction equation $\partial K/\partial x = \partial^2 K/\partial y^2$ and it is a self-similar solution of the form $K(x, y) = x^{1/4} G(y/2\sqrt{x})$. Hence an ordinary differential equation satisfied by the function $G(\nu)$, $\nu = y/2\sqrt{x}$ is found:

$$G''_{\nu\nu} + 2\nu G'_\nu = G, \quad G(0) = 1/\Gamma(5/4), \quad G'(0) = -2/\Gamma(3/4). \tag{15}$$

The solution of Eq. (15) may be expressed in terms of a degenerate hypergeometric function of the first kind $F(\alpha, \beta, z)$ [9]

$$G(\nu) = F\left(-\frac{1}{4}, \frac{1}{2}, -\nu^2\right) \Big/ \Gamma(5/4) \; - 2\nu F\left(\frac{1}{4}, \frac{3}{2}, -\nu^2\right) \Big/ \Gamma(3/4). \tag{16}$$

Using well-known [9] expansions for degenerate hypergeometric functions, the result obtained is

$$G(\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n (2\nu)^n}{n!\;\Gamma(5/4 - n/2)} \sim 2\sqrt{\pi} \exp(-\nu^2)/\nu^{3/2}, \tag{17}$$

where the series converges at all ν and is convenient at small ν and the second relation is an asymptotic formula suitable for calculations as $\nu \to \infty$ (Fig. 1, curve 1). The expression for Q may now be written in the form

$$Q = \pi^{1/2} \xi^{1/4} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{G[T(4n+1)/2\sqrt{\xi}]}{4^n}. \tag{18}$$

Using asymptotic Eq. (17) and the Stirling formula, it may be shown that the series in Eq. (18) converges when $T/\sqrt{\xi} > 0$. As $T/\sqrt{\xi} \to \infty$, leaving one term in the series, an asymptotic formula is obtained

$$Q \sim 4\sqrt{2}\,\pi\xi \exp(-T^2/4\xi)/T^{3/2}, \quad T^2/\xi \to \infty. \tag{19}$$

The expansion in Eq. (18) is convenient for calculations at large $T/\sqrt{\xi}$. Using Eq. (13), a theoretical relation is also obtained for small $T/\sqrt{\xi}$.

The singular points of the function Q^* are branch points $p_k = -(\pi k)^2/4T^2$, $k = 0, 1, 2, \ldots$; the function Q^* is unique in the plane cut along segments of the real axis connecting the points p_0 and p_1, p_2 and p_3, etc. (Fig. 2). The Cauchy theorem is applied to the integral of the function $Q^* \exp(p\xi)$ over the contour in Fig. 2. The circle of large radius R_N intersects the real axis at the points $R_N = -(\pi/2 + 2\pi N)^2/4T^2$, and $|\operatorname{sh}^{-1/2}(2T\sqrt{p})| \leq 1$ over the whole line γR_N; therefore, taking account of the factor $p^{-5/4}$, the Jordan lemma may be applied to the integral over the system of contours γR_N as $N \to \infty$. For all points p_k, except p_0, the integrals over the small circles $\gamma\rho_k$ as $\rho_k \to 0$ $(k = 1, 2, 3, \ldots)$ also tend to zero; therefore passing to the limit as $N \to \infty$, $\rho_k \to 0$ in the Cauchy and Riemann-Mellin theorems leads to the expression $(\zeta = \xi/4T^2)$

$$\frac{Q}{\sqrt{\pi T}} = \frac{1}{2\pi i} \int_\varkappa \frac{\exp(p\zeta)\,dp}{p^{5/4}\sqrt{\operatorname{sh}\sqrt{p}}} + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \int_{(2\pi k)^2}^{(2k+1)^2\pi^2} \exp(-r\zeta)\,dr/r^{5/4} \quad \sqrt{\sin\sqrt{r}}, \tag{20}$$

where $\varkappa$ is a loop covering the points $p = 0$ and $p = -\pi^2$ (Fig. 2). At large ζ, Laplace asymptotic method [10] may be applied to the integral in the summation; it is then found that these integrals are exponentially small as $\zeta \to \infty$; this indicates good convergence of the series in Eq. (20). Therefore, the basic (algebraic) contribution to the asymptotic expansion of Q

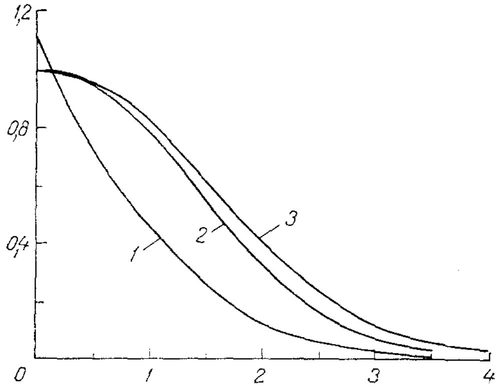

Fig. 1. Graph of the functions: 1) $G = F(y/\sqrt{x})$ according to Eq. (17); 2) $Q\sqrt{T/\xi} = F(T/\sqrt{\xi})$; 3) $4Q_1T_1\sqrt{u}/\pi C_0 hm\sqrt{D} = F(2T_1/\sqrt{m})$; the first quantity is plotted along the ordinate and the second (in parentheses) along the abscissa. All the quantities are dimensionless.

will be given by the vicinity of the point p = 0. Suppose that b_k, k = 0, 1, 2, ..., are coefficients of the expansion in Maclaurin series of the analytical function $[\sqrt{p}/\mathrm{sh}\sqrt{p}]^{1/2}$ ($b_0 = 1$, $b_1 = -1/12$, and so on). The integration contour $\varkappa$ is replaced by $\varkappa_1$, departing to an infinitely remote point over both sides of the negative real semiaxis of the plane p. The distortion introduced here is negligibly small. Using the well-known Hankel formula [10] for the function $1/\Gamma(z)$, an asymptotic expansion is obtained

$$\frac{Q}{\sqrt{\pi T}} \sim \sum_{k=0}^{\infty} \frac{b_k}{2\pi i} \int_{\varkappa_1} \frac{\exp(p\zeta)}{p^{3/2-k}}\, dp = \sum_{k=0}^{\infty} b_k \zeta^{1/2-k}/\Gamma(3/2-k). \tag{21}$$

It is evident from Eq. (21) that $Q \to \infty$ as $T \to 0$ ($\xi \neq 0$), as would be expected on the basis of Eq. (13). This is because the integral with respect to η is taken with an infinite upper limit, in order to simplify the calculation of Q; with the initial condition in Eq. (9), this must lead to an infinite value of Q. In reality, the upper limit is large (of the order of $\varepsilon^{-1/4}$) but not infinite. This simplification becomes more accurate as T increases. This is one more reason why the solution is applicable at sufficiently large times.

II. Channel Bounded by a Closed Curve of Arbitrary Form

As is known, the velocity profile of liquid flowing in a prismatic channel is described by the solution of the equation $\Delta v = -P/\ell\mu$ with zero boundary condition at the contour. The function v is superharmonic and therefore must exceed the solution of the Laplace equation for the same region and the same boundary condition, i.e., $v \geqq 0$ everywhere in the channel. The case $v \equiv 0$ is impossible. From the Weierstrass theorem, it follows that the function v attains a maximum at some internal point. If there are several points of maximum v, the liquid flow in the vicinity of one of these is considered.

The velocity profile in the vicinity of the maximum point is determined in Cartesian coordinates of negative-definite quadratic form, which in the principal axes may be written in the form $v = u(1 - x^2/h_1^2 - y^2/h_2^2)$, where the x and y axes are orthogonal. As before, passing to a moving coordinate system, the following internal equation is obtained for the impurity concentration

$$\partial c/\partial t + u(x^2/h_1^2 + y^2/h_2^2)\,\partial c/\partial m = D(\partial^2 c/\partial x^2 + \partial^2 c/\partial y^2). \tag{22}$$

The external solution has a structure similar to Eq. (7); therefore, the following additional conditions apply

$$c|_{t=0} = 1,\quad c|_{m=0} = 0,\quad c|_{x,y\to\infty} \to 0;\quad \partial c/\partial x|_{x=0} = 0,\quad \partial c/\partial y|_{y=0} = 0. \tag{23}$$

Equation (22), describing the impurity propagation close to the tube axis, may also be applied in more complex cases, i.e., with additional conditions not in the special form of Eq. (23) but with some number of arbitrary functions. Note that, with usual conditions of the first, second, and third kind at the planes x = 0 and y = 0, and inhomogeneous conditions when m = 0 and t = 0, the solution of Eq. (22) is found by the method of [11]. Laplace transformation with respect to the variable m is first applied, and then transformation with

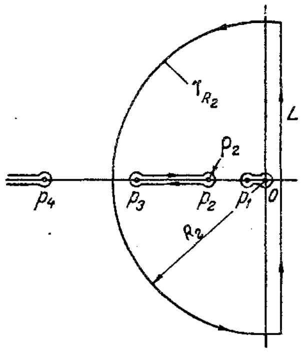

Fig. 2. Integration contour.

respect to the eigenfunctions of the equation $DX'' = (ux^2/h_1^2 - \lambda_n)X$ with the corresponding boundary conditions, where the transformation with respect to the variables x and y may be taken in any sequence. The resulting ordinary differential equation in terms of time-dependent functions is easily solved, and inverse integral transformations are then undertaken.

It follows from the first three conditions in Eq. (23) that the solution of Eq. (22) is an even function of the variables x and y. Therefore, the last two conditions in Eq. (23) are actually a consequence of the first three and the structure of Eq. (22), i.e., a region $x, y \in [0, \infty)$ may be considered, as in the plane problem.

Applying a Laplace transformation with respect to m to Eq. (22) gives

$$\partial c^*/\partial t + u(x^2/h_1^2 + y^2/h_2^2)\, pc^* = D(\partial^2 c^*/\partial x^2 + \partial^2 c^*/\partial y^2). \tag{24}$$

With the additional conditions in Eq. (23), as a result of direct verification, it may be established that the solution of Eq. (24) is $c^* = \Phi^*(x, t)\Phi^*(y, t)$, where Φ^* satisfies (with insignificant scale changes) the problem in Eqs. (10) and (11) for the plane case. Therefore, the final results of Sec. I may be used and formulas corresponding to Eq. (13) may be introduced at once

$$Q_1 = \int_0^\infty \int_0^\infty c^* dx dy = \frac{\pi (Dh_1h_2/u)^{1/2}}{2p^{3/2}[\mathrm{sh}(2T_1\sqrt{p})\,\mathrm{sh}(2T_2\sqrt{p})]^{1/2}} = \frac{\pi (Dh_1h_2)^{1/2}}{\sqrt{up^3}} \times$$
$$\times \sum_{n,k=0}^{\infty} \frac{(2n)!\,(2k)!}{(n!)^2\,(k!)^2\,4^{n+k}} \exp\{-[T_1(4n+1) + T_2(4k+1)]\sqrt{p}\}, \tag{25}$$
$$T_{1,2} = \frac{t(Du)^{1/2}}{h_{1,2}}.$$

It is known that

$$2\sqrt{x}\,\mathrm{ierfc}\left(\frac{y}{2\sqrt{x}}\right) = 2\left(\frac{x}{\pi}\right)^{1/2} \exp\left(-\frac{y^2}{4x}\right) - y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{x}}\right) =$$
$$= \frac{1}{2\pi i}\int_L \frac{\exp(px - y\sqrt{p})}{p^{3/2}}\, dp, \tag{26}$$

where $\mathrm{erfc}(z) = (2/\sqrt{\pi})\int_z^\infty \exp(-t^2)\,dt$ is an additional probability integral. After term-by-term manipulation of the series in Eq. (25), the following expression is obtained

$$Q_1 = \frac{2\pi (Dmh_1h_2)^{1/2}}{u^{1/2}} \sum_{n,k=0}^{\infty} \frac{(2n)!\,(2k)!}{(n!)^2\,(k!)^2\,4^{n+k}}\, \mathrm{ierfc}\{[T_1(4n+1) + T_2(4k+1)]/2\sqrt{m}\}. \tag{27}$$

Retaining the first term at large $T_{1,2}/\sqrt{m}$ and using the asymptotic formula for the probability integral [9], the following asymptotic relation is obtained for Q_1:

$$Q_1 \sim \frac{4(\pi Dh_1h_2m^3)^{1/2}}{u^{1/2}(T_1+T_2)^2} \exp[-(T_1+T_2)^2/4m], \quad \frac{T_1+T_2}{\sqrt{m}} \to \infty. \tag{28}$$

The behavior of Q_1 as $T_{1,2}/\sqrt{m} \to 0$ is determined by the residue of $Q_1^* \exp(mp)$ at the point $p = 0$ and takes the form

$$Q_1 = \pi h_1 h_2 [m - (T_1^2 + T_2^2)/3]/2ut, \quad T_{1,2}^2/m \to 0. \tag{29}$$

The particular case $h_1 = h_2 = h$ is of interest; it is realized, for example, in a circular channel and a channel bounded by the sides of a regular polygon. Here $T_1 = T_2$ and Eq. (25) may be represented by a series

$$Q_1^*/\pi h (D/u)^{1/2} = 0{,}5/p^{3/2}\,\mathrm{sh}(2T_1\sqrt{p}) = \sum_{n=0}^{\infty} \exp[-2T_1(2n+1)\sqrt{p}]/p^{3/2}, \tag{30}$$

manipulation of which gives the expression

$$Q_1 = 2\pi h (Dm/u)^{1/2} \sum_{n=0}^{\infty} \mathrm{ierfc}\,[T_1(2n+1)/\sqrt{m}], \tag{31}$$

which is convenient at large $T_1/\sqrt{m}$. By the usual methods, applying the residue theorem to the integral of the function $Q_1^* \exp(mp)$ over the contour L, R_N ($N \to \infty$) in Fig. 2 (the singular points are the same as in the case of Eq. (13) except that now they are poles), the following expression may be obtained

$$Q_1/\pi h T_1 (D/u)^{1/2} = \frac{m}{4T_1} - \frac{1}{6} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi^2 k^2 m}{4T_1^2}\right], \tag{32}$$

which is convenient for calculations at small $T_1/\sqrt{m}$. The results of calculations by Eqs. (31) and (32) are illustrated in Fig. 1 (curve 3).

Remember that in view of the symmetry only a quarter of the flow region close to the tube axis is considered here. To obtain the total amount of impurity in the channel, the right-hand side of Eqs. (27)-(29), (31), and (32) must be multiplied by four.

III. Determining the Constants $h_{1,2}$ and the Maximum Velocity u

To employ the relations obtained in Sec. II, three parameters of the velocity profile must be determined in the vicinity of the maximum-velocity point: $h_{1,2}$ and u. Equations may be obtained for determining these quantities under the condition that the function performing conformal mapping of the region bounded by the channel contour onto a unit circle is known: $z = f(\zeta)$, $z = x + iy$, $\zeta = \xi + i\eta$. Without loss of generality, the constant right-hand side of the Poisson equation for the function may be regarded as equal to -2. The function v is sought in the form $v = w - x^2 - y^2$, where the function w is harmonic and satisfies the condition $w = x^2 + y^2 = |f^2|$ at the channel contour (a unit circle in the ζ plane). As is known, conformal mapping of singly connected regions is determined with an accuracy of up to three real parameters. In the present case, a general formula of the form $f(r)$, $r - \exp(i\psi)(\zeta - a)/(1 - \bar{a}\zeta)$ may be obtained from the function f, where the complex number $a = \alpha + i\beta$ [10] determines the coordinate of the center of the circle of plane r in plane ζ. The third parameter ψ determines the rotation of the circle. It is significant in choosing the principal axes of the quadratic form of the velocity profile; then $\psi = 0$ is assumed, and the function w in the plane ζ is assumed to depend on the two real parameters α and β, which are ordered so that some point of velocity maximum falls at the coordinate origin of the ζ plane. The value of w at the contour of the circle is denoted by $\omega(\alpha, \beta, \varphi)$, where φ is the polar angle. Then the Fourier method leads to the result

$$v = -|f|^2 + a_0 + a_1\xi + d_1\eta + \sum_{n=2}^{\infty} r^n [a_n \cos(n\varphi) + d_n \sin(n\varphi)], \tag{33}$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \omega(\alpha, \beta, \varphi)\, d\varphi; \quad a_n = \frac{1}{\pi} \int_0^{2\pi} \omega(\alpha, \beta, \varphi) \cos(n\varphi)\, d\varphi;$$

$$d_n = \frac{1}{\pi} \int_0^{2\pi} \omega(\alpha, \beta, \varphi) \sin(n\varphi)\, d\varphi, \quad n = 1, 2, 3, \ldots; \quad r = (\xi^2 + \eta^2)^{1/2}.$$

Suppose that the expansion of $|f|^2$ in Maclaurin series takes the form

$$|f|^2 = l_{00} + \xi l_{10} + \eta l_{01} + \xi^2 l_{20} + 2\xi\eta l_{11} + \eta^2 l_{02} + \ldots, \quad (34)$$

where all the functions ℓ_{ij}, a_i, and d_i depend on α and β. Then, choosing α and β such that the conditions

$$a_1 = l_{10}, \quad d_1 = l_{01} \quad (35)$$

are satisfied, it may be arranged that the point $\xi = 0$, $\eta = 0$ is stationary for the function v. The quadratic form for the velocity profile is now determined by the expression

$$\xi^2 (a_2 - l_{20}) - \eta^2 (a_2 + l_{02}) + 2\xi\eta (d_2 - l_{11}), \quad (36)$$

which must be divided by $|f'(0)|^2$ in the plane z on passing to the coordinates x and y in the vicinity of the maximum-velocity point, in view of the conformity of the mapping. The choice of principal axes in the z plane and the determination of $h_{1,2}$ from the expression obtained are not difficult. The existence of a solution of Eq. (35) follows from general considerations; see Sec. II. Note, however, that it is not necessarily unique and in addition the quadratic form of Eq. (36) obtained may be indeterminate. This may be understood intuitively if the channel profile is taken in the form of two identical circles connected by a narrow neck, so that the whole figure is symmetric. Then, approximately at the center of each circle, there is a velocity maximum (nonuniqueness) and at the center of the whole figure there is a saddle point (indefinite quadratic point). For channels where the velocity profile is unknown (circle, ellipse, equilateral triangle, etc.), determining the parameters $h_{1,2}$ and u is simpler and often does not pose any difficulties. Note also that, in view of the application of the equation $\Delta v = -2$ to a broad range of elasticity-theory problems (rod torsion, membrane flexure) and in other fields, Eq. (35) and the other results of Sec. III are of definite importance also in these areas of mathematical physics.

The value of u is found from Eq. (33) by the substitution $\xi = 0$, $\eta = 0$:

$$u = (a_0 - l_{00})\, P/2l\mu, \quad (37)$$

where the values of α and β found from Eq. (35) must be used.

As an example of finding $h_{1,2}$ and u, consider liquid flow in a region bounded by a single loop of the Bernoulli lemniscate, defined by the equation: $(x^2 + y^2)^2 = 2(x^2 - y^2)$. As is known [10], a branch of the lemniscate is mapped onto a circle of unit radius by the function $z = \sqrt{\zeta + 1}$. In the view of the lemniscate symmetry with respect to the y axis, it is clear that the maximum-velocity point has the coordinate y = 0 and the parameter β must be set equal to zero. Thus, the mapping function must be taken in the form

$$z = [(1-\alpha)(\zeta + 1)/(1 - \alpha\zeta)]^{1/2}, \quad \alpha \in [0, 1), \quad (38)$$

and hence it is simple to find the function $\omega(\alpha, 0, \varphi) = 2(1-\alpha)|\cos(\varphi/2)|/(1 - 2\alpha\cos\varphi + \alpha^2)^{1/2}$. In addition, it is known that $d_i = 0$, $i = 1, 2, 3, \ldots$; $\ell_{00} = 1 - \alpha$; $\ell_{10} = 1 - \alpha^2$; $\ell_{01} = 0$; $\ell_{20} = 1.5\alpha^2(1 - \alpha)$; $\ell_{11} = 0$; $\ell_{02} = 0.5(1 + \alpha)(1 - \alpha)^2$. Calculating the integral a_1, an equation for determining α is obtained from the first relation in Eq. (35) (the second is satisfied identically 0 = 0):

$$\frac{\pi\alpha(1+\alpha)}{2(1-\alpha)} = \left(R + \frac{1}{R}\right)\ln(R + \sqrt{R^2+1}) - \sqrt{R^2+1}, \quad R = \frac{2\sqrt{\alpha}}{1-\alpha}, \quad (39)$$

which has the single root $\alpha = 0.3430$ in the interval $\alpha \in [0, 1)$. The integrals a_0 and a_2 are also calculated analytically and, when $\alpha = 0.3420$, take the values: $a_0 = 0.9584$; $a_2 = -4.6054$. Hence using Eq. (37), the maximum value $u = 0.1507P/\ell\mu$ is determined and then, from Eq. (36) – the form of Eq. (36) has already been obtained in the principal axes – the parameters h_1 and h_2 are found: $h_1 = 0.2620$; $h_2 = 0.2505$. The coordinate of the maximum-velocity point is found from Eq. (38) by substitution of the condition $\zeta = 0$: x = 0.8105. The parameter values obtained may easily be recalculated for a lemniscate of more general form including some dimensional parameter. It is noteworthy that the desired parameters are dimensional in the given example.

Note, in conclusion, that, although additional conditions in Eq. (5) of particular form have been used, the actual solution of the internal problem is more general. For example, in the case of the general condition $c|_{z=0} = c_*(y, \tau)$, the result obtained for a function $c_*(y, \tau)$ of sufficiently broad form in the internal variables is: $c|_{z=0} = c_*(0, \infty) = \text{const}$, which coincides with the corresponding condition in Eq. (5).

NOTATION

C, c, dimensional and dimensionless impurity concentrations; C_0, initial concentration; D, impurity-diffusion coefficient; L, γ_R, γ_ρ, parts of integration contour; ℓ_{ij}, a_i, d_i, coefficients in the expansion of the functions $w = v + x^2 + y^2$ and $|f|^2$ in the corresponding series of Eqs. (33) and (34); m = ut − z, dimensional coordinate of the moving coordinate system; h, halfwidth of the channel in plane problem; h_1, h_2, coefficients in the quadratic form of the velocity profile; Q, amount of impurity in some cross section of the channel per unit area; Q_1, amount of impurity per unit length in the three-dimensional case; P/ℓ, pressure difference per unit length of channel; p, Laplace-transformation variable; T, η, internal variables; t, time; u, maximum value of the velocity; Z, Y, Cartesian coordinates longitudinal and transverse to the flow; v, liquid velocity in channel; $\xi = \tau - z$, variable in coordinate system moving at velocity u; μ, viscosity of liquid; Γ(x), Euler gamma function.

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SOLUTION OF THE CONVERSE THERMAL CONDUCTIVITY PROBLEM WITH CONSIDERATION OF THE PERTURBING INFLUENCE OF THE THERMOCOUPLE

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Questions involving the use of an adequate model for temperature measurement in solving converse thermal conductivity problems are considered.

Methods for solution of converse thermal conductivity problems are one of the most promising means for adequate processing of data in thermophysical experiment. At the present time a number of highly effective methods have been developed for solution of such problems [1], although the majority of these can be used only under conditions where the perturbing action of thermocouples on heat propagation in the body under study can be neglected. In many cases of practical importance the effect of thermocouples is quite significant [2-5].

In particular, this is true in the study of processes of casting metals and alloys or in temperature measurements in a cutting instrument where the dimensions of the thermocouple, its insulation, and the channel in which these are located are comparable to the distances to the heat source and the area of the surface upon which it acts. In such cases the temperature sensor must be considered as an independent body with its own thermophysical and geometric characteristics, actively participating in heat exchange with the surrounding object.

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